

Sampling and Aliasing Properties of Three-Dimensional Point-Lattices

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Abstract—Sampling properties of some three-dimensional lattices are examined. The Cartesian cubic lattice is compared with the non-Cartesian face- and body-centered cubic lattices. Numerical computations using analytical functions are used to compare rotational dependency and aliasing effects.

I. INTRODUCTION

There are many imaging techniques that generate three-dimensional volume images today. With higher precision in the image acquisition equipment, storing and processing these images require an increasing amount of data processing capacity. Traditionally, three-dimensional images are represented by cubic (or cuboid) picture elements on a cubic lattice¹.

In this report, sampling and aliasing properties of some three-dimensional lattices are presented. The use of non-standard lattices, such as the 2D hexagonal lattice, in image analysis, image processing, and computer graphics is often motivated by the high sampling efficiency of these lattices [1]–[3]. A lattice is generated by a matrix $\mathbf{V}$ is obtained by linear integer combinations of the columns of $\mathbf{V}$.

In the one-dimensional case, given a function f bandlimited by B (the Fourier transform of f , F is such that $|F(u)| = 0$ if $|u| > B$), the *optimal* sampling is to sample the function slightly below $\frac{1}{2B}$, i.e., $\Delta x = \frac{1}{2B} - \epsilon$. This is the well known Shannon/Nyquist sampling theorem. In higher dimensions, given an isotropic function f bandlimited by B , F is also isotropic which implies that it is non-zero only in the interior of a ball of radius B . Sampling f on points according to a sampling lattice generated by $\mathbf{V}$ gives spectra replicated according to the inverse of the transpose of $\mathbf{V}$, i.e., $\mathbf{V}^{-T} = (\mathbf{V}^{-1})^T$. The lattice is optimal in frequency domain if the balls of radius B that contain the spectrum are as densely packed as possible. Thus, a dense lattice in frequency domain is needed. The densest point-lattice in three dimensions is the face-centered cubic (fcc) lattice [4]. If a point-lattice generated by $\mathbf{V}$ is such that $\mathbf{V}^{-T}$ is an fcc lattice, then the lattice generated by $\mathbf{V}$ is a body-centered cubic (bcc) lattice [2], [3], [5].

The Voronoi regions of the cubic, fcc, and bcc lattices are shown in Figure 1.

In this report, sampling properties of the cubic, fcc, and bcc lattices are compared. The energy that is represented

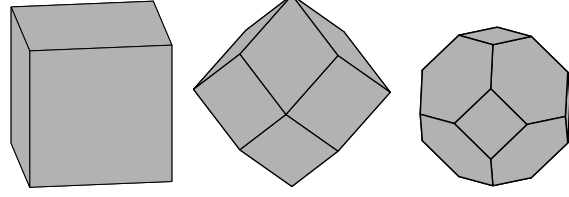


Fig. 1. Voronoi regions (voxels) in (from left to right) the cubic, fcc, and bcc lattices.

by sampling a function is analyzed in frequency domain. By using Plancherel theorem, the energy can be calculated by the energy in one period in frequency domain. Rotational dependency will be compared by using function values on a one-dimensional linear subsets in spatial domain. Also, results on sampling a three-dimensional version of the Shepp-Logan phantom are presented.

II. (DISCRETE SPACE) FOURIER TRANSFORM

The following definition of the Fourier transform $\mathcal{F}(f(\mathbf{x}))$ of a function $f(\mathbf{x})$ will be used

$$F(\mathbf{u}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \mathbf{u}^T \mathbf{x}} d\mathbf{x} \text{ for any } \mathbf{u} \in \mathbb{R}^n. \quad (1)$$

The inverse Fourier transform $\mathcal{F}^{-1}(F(\mathbf{u}))$ is

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} F(\mathbf{u}) e^{2\pi i \mathbf{x}^T \mathbf{u}} d\mathbf{u} \text{ for any } \mathbf{x} \in \mathbb{R}^n.$$

Change of variables in spatial domain gives

$$\begin{aligned} \mathcal{F}(f(\mathbf{V}\mathbf{t})) &= \int_{\mathbb{R}^n} f(\mathbf{V}\mathbf{t}) e^{-2\pi i \mathbf{u}^T \mathbf{t}} d\mathbf{t} \\ &= \frac{1}{|\det \mathbf{V}|} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \mathbf{u}^T \mathbf{V}^{-1} \mathbf{x}} d\mathbf{x} \\ &= \frac{1}{|\det \mathbf{V}|} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \mathbf{u}^T ((\mathbf{V}^{-1})^T)^T \mathbf{x}} d\mathbf{x} \\ &= \frac{1}{|\det \mathbf{V}|} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i ((\mathbf{V}^{-1})^T \mathbf{u})^T \mathbf{x}} d\mathbf{x} \\ &= \frac{1}{|\det \mathbf{V}|} F(\mathbf{V}^{-T} \mathbf{u}). \end{aligned}$$

In other words, we have the following Fourier pair:

$$f(\mathbf{V}\mathbf{x}) \longleftrightarrow \frac{1}{|\det \mathbf{V}|} F(\mathbf{V}^{-T} \mathbf{u}).$$

The Dirac comb, or Shah function, is

$$\text{III}_{\mathbf{V}}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^n} \delta(\mathbf{x} - \mathbf{V}\mathbf{n}),$$

¹A (point) lattice is a discrete subgroup of Euclidean space, which contains the origin.

where $\delta(\mathbf{x})$ is the n -dimensional Dirac delta function. The Fourier transform of the Dirac comb is

$$\sum_{\mathbf{n} \in \mathbb{Z}^n} \delta(\mathbf{x} - \mathbf{V}\mathbf{n}) \longleftrightarrow \frac{1}{|\det(\mathbf{V})|} \sum_{\mathbf{n} \in \mathbb{Z}^n} \delta(\mathbf{u} - \mathbf{V}^{-T}\mathbf{n}).$$

Convolution is denoted by $*$, i.e.,

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{t})g(\mathbf{x} - \mathbf{t})d\mathbf{t}$$

The convolution theorem yields

$$f(\mathbf{x})\text{III}_{\mathbf{V}}(\mathbf{x}) \longleftrightarrow \frac{1}{|\det(\mathbf{V})|} (F(\mathbf{u}) * \text{III}_{\mathbf{V}^{-T}}(\mathbf{u})).$$

Let $\Pi_{\mathbf{V}}$ be the Voronoi region of the lattice point $\mathbf{0}$ in the lattice defined by $\mathbf{V}$. The Voronoi region on the cubic, fcc, and bcc lattices are cubes, rhombic dodecahedra, and truncated octahedra, respectively, [1]–[3], [5]. Let also

$$\Pi_{\mathbf{V}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Pi_{\mathbf{V}} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi_{\mathbf{V}}$ be the ideal lowpass function defined as:

$$\mathcal{F}(\phi_{\mathbf{V}}(\mathbf{x})) = \frac{1}{|\det(\mathbf{V})|} \Pi_{\mathbf{V}^{-T}}(\mathbf{u}). \quad (2)$$

If f is bandlimited (i.e., F is zero outside $\Pi_{\mathbf{V}^{-T}}$), then f is completely represented by its samples since

$$\begin{aligned} f(\mathbf{x}) &= \int_{\mathbb{R}^n} F(\mathbf{u})e^{2\pi i\mathbf{u}^T\mathbf{x}}d\mathbf{u} = \\ \int_{\Pi_{\mathbf{V}^{-T}}} F(\mathbf{u})e^{2\pi i\mathbf{u}^T\mathbf{x}}d\mathbf{u} &= (f(\mathbf{x})\text{III}_{\mathbf{V}}(\mathbf{x})) * \phi_{\mathbf{V}}(\mathbf{x}). \end{aligned} \quad (3)$$

III. SAMPLING AND ALIASING

Let $g(\mathbf{x}) = f(\mathbf{x}) * \phi_{\mathbf{V}}(\mathbf{x})$, i.e., the reconstructed function from the samples of f using (3). As performance measure, the *aliasing error*, $\mathcal{E}$, (the relative error of the energy that is represented by the function reconstructed from the samples, see [6]), is used. The Plancherel theorem (the L^2 norm is preserved by the Fourier transform) is used to get

$$\begin{aligned} \mathcal{E} &= 1 - \frac{\int_{\mathbb{R}^n} |g(\mathbf{x})|^2 d\mathbf{x}}{\int_{\mathbb{R}^n} |f(\mathbf{x})|^2 d\mathbf{x}} = 1 - \frac{\int_{\mathbb{R}^n} |\mathcal{F}(g(\mathbf{x}))|^2 d\mathbf{u}}{\int_{\mathbb{R}^n} |\mathcal{F}(f(\mathbf{x}))|^2 d\mathbf{u}} \\ &= 1 - \frac{\int_{\Pi_{\mathbf{V}^{-T}}} |F(\mathbf{u})|^2 d\mathbf{u}}{\int_{\mathbb{R}^n} |F(\mathbf{u})|^2 d\mathbf{u}}. \end{aligned} \quad (4)$$

The following Gauss function will be used as test function

$$\begin{aligned} f(x, y, z) &= e^{-(x^2+y^2+z^2)} \text{ with} \\ \mathcal{F}(f(x, y, z)) &= F(u, v, w) = \pi^{(3/2)} e^{-\pi^2(u^2+v^2+w^2)}. \end{aligned}$$

The total energy is

$$\begin{aligned} \int_{\mathbb{R}^n} \left(e^{-(x^2+y^2+z^2)} \right)^2 dx dy dz &= \\ \int_{\mathbb{R}^n} \pi^3 \left(e^{-\pi^2(u^2+v^2+w^2)} \right)^2 du dv dw &= \\ \frac{\sqrt{2}}{4} \pi^{(3/2)} &\approx 1.97. \end{aligned}$$

The effect of sampling is simulated by using (4). Cubic, fcc, and bcc sampling lattices in with equal sampling density will be compared by considering reciprocal lattices with equal volume of the Voronoi regions. The plot in Figure 2 shows the aliasing error (y-axis) that is covered by sampling using sampling lattices of increasing density (x-axis).

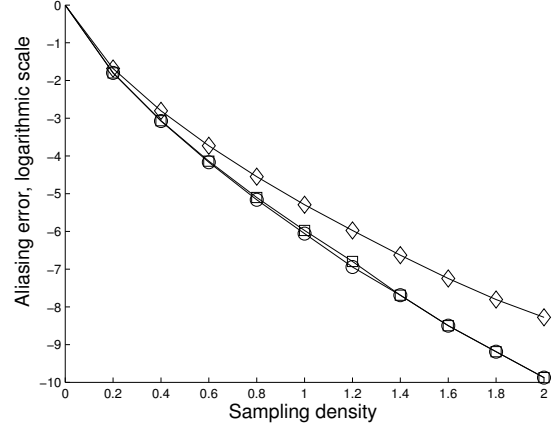


Fig. 2. The aliasing error as a function of sampling density when sampling $f(x, y, z) = e^{-(x^2+y^2+z^2)}$ on a cubic (◇), fcc (□), and bcc (○) lattice.

For the test function, numerical experiments show the sampling that covers 98% of the relative energy by adjusting the density of the sampling lattice in (4). Compare with the plot in Figure 2. The volume of $\Pi_{\mathbf{V}^{-T}}$, i.e., the period in frequency domain, needed to cover 98% of the relative energy for the cubic, fcc, and bcc lattices are 0.64, 0.56, and 0.55, respectively. This corresponds to sampling densities 1.55, 1.80, and 1.81, respectively, in spatial domain.

Now the sampling density in spatial domain is set to 1.55, the relative energy covered by the sampling in the cubic lattice is 98%. The corresponding values for the fcc and bcc lattices are 98.7% and 98.8%, respectively.

A. Rotational dependency

For a number of rotations, a 1D subset in spatial domain is analyzed. We describe the formula when the 1D subset is aligned with the x -axis. The rotation is obtained by rotating the domain of integration.

Analogous to the Fourier slice theorem, a slice in the spatial domain corresponds to a projection in frequency domain:

$$f(x, 0, 0) \longleftrightarrow \int_{\mathbb{R}^2} F(u, v, w) dv dw. \quad (5)$$

The total energy in the 1D subset of a separable function $f(x, y, z) = f(x)f(y)f(z)$ with real Fourier transform F is

$$\begin{aligned} \int_{\mathbb{R}} (f(x)f(0)f(0))^2 dx &= \\ \int_{\mathbb{R}} F(u)^2 \left(\int_{\mathbb{R}^2} F(v)F(w)dv dw \right)^2 du. \end{aligned}$$

The reconstructed function is now analyzed, i.e., the formula in (3). For the analysis in frequency domain, (5) will be used.

Given C , a compact set in $\mathbb{R}^3$, let $W_1(u, v)$, $W_2(u, v)$, $V_1(u)$, $V_2(u)$, U_1 , U_2 bound the set, i.e.,

$$C = \left\{ (u, v, w) : W_1(u, v) \leq w \leq W_2(u, v), \right. \\ \left. V_1(u) \leq u \leq V_2(u), U_1 \leq u \leq U_2. \right\}.$$

Rotated versions of $\prod_{\mathbf{v}-\tau}$ are used as domain C . Remember that the cubic, fcc, and bcc lattices correspond to the Voronoi region in the reciprocal lattice (Brillouin zones) in frequency domain, i.e., cubic, truncated octahedral, and rhombic dodecahedral domain, respectively.

The effect of sampling a one-dimensional linear subset on the three lattices is now considered. The test function will be used. In general, the aliasing error along one-dimensional structures in a 3D volume is simulated by this procedure.

Consider

$$f(x, y, z) = e^{-(x^2+y^2+z^2)} \text{ with } z = 0, y = 0.$$

The Fourier transform of $f(x) = e^{-x^2}$ is $\sqrt{\pi}e^{-\pi^2 u^2}$ and the total energy is $\sqrt{2\pi}/2$. By the Fourier slice theorem, the energy in the 1D profile is

$$\int_{U_1}^{U_2} F(u)^2 \left(\int_{V_1(u)}^{V_2(u)} \int_{W_1(u,v)}^{W_2(u,v)} F(v)F(w) dw dv \right)^2 du. \quad (6)$$

For the numerical computations, the integral is rewritten using Fubini's theorem and get

$$\int_{U_1}^{U_2} F(u)^2 \left(\int_{V_1(u)}^{V_2(u)} \int_{W_1(u,v)}^{W_2(u,v)} F(v)F(w) dw dv \right) \cdot \\ \left(\int_{V_1(u)}^{V_2(u)} \int_{W_1(u,v')}^{W_2(u,v')} F(v')F(w') dw' dv' \right) du = \\ \int_{U_1}^{U_2} \int_{V_1(u)}^{V_2(u)} \int_{W_1(u,v)}^{W_2(u,v)} \int_{V_1(u)}^{V_2(u)} \int_{W_1(u,v')}^{W_2(u,v')} F(u)^2 \cdot \\ F(v)F(w)F(v')F(w') dw' dv' dw dv du = \\ \int_{(u,v,w) \in C \text{ and } (u,v',w') \in C} F(u)^2 \cdot \\ F(v)F(w)F(v')F(w') dw' dv' dw dv du$$

This integral can be computed numerically by standard methods. A Monte-Carlo method with bounds on the estimated error is used.

In Figure 3, the aliasing error $1 - I/(\sqrt{2\pi}/2)$, where $\sqrt{2\pi}/2$ is the total energy and I is the integral (6), for some different rotations are shown.

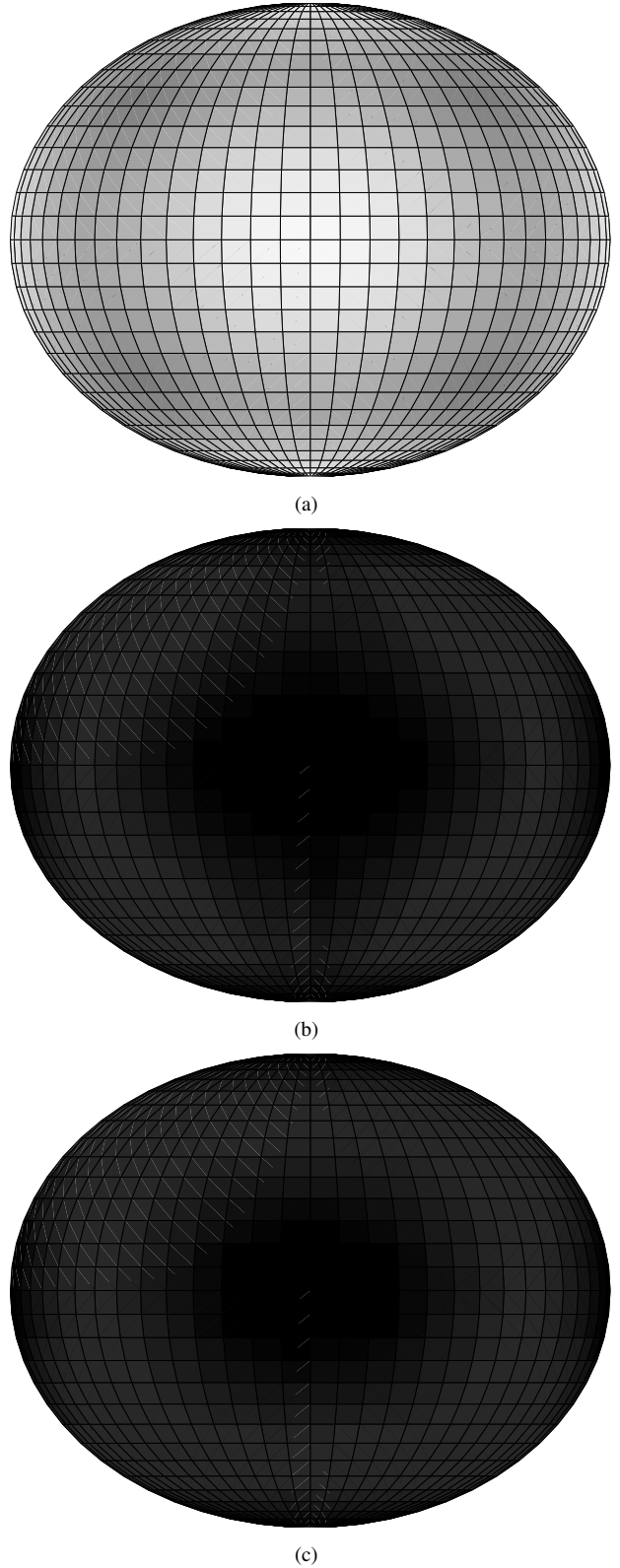


Fig. 3. Aliasing error of a 1D profile as function of direction. The sampling density 1.55 is used for all lattices. The spheres are viewed from the y -direction and the z -direction is up. A linear scale between 16.2% (black) and 21.8% (white) is used. Minimum/maximum error is 18.6%/21.1% (cubic lattice, a), 16.2%/17.0% (bcc lattice, b), and 16.2%/17.0% (fcc lattice, c),



Fig. 4. A three-dimensional version of the Shepp-Logan phantom.

B. Sampling an analytical 3D Phantom

In this section, some experiments on a three-dimensional version of the Shepp-Logan phantom, see Figure 4, are performed. The 3D Shepp-Logan phantom is composed of the union of a number of ellipsoids. Since the Fourier transform of a general ellipsoid is known, the Fourier transform of the 3D Shepp Logan can be calculated, [7].

The energy represented by sampling at a given density is again simulated by integration in frequency domain. In Figure 5, the energy in that is represented by sampling the 3D phantom independent of rotation of the sampling grid is given as a function of sampling density.

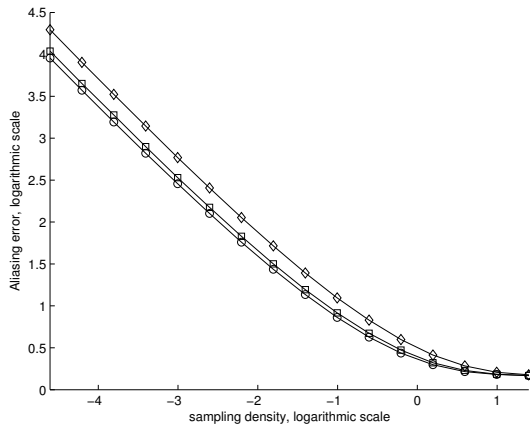


Fig. 5. The aliasing error as a function of sampling density when sampling the 3D Shepp Logan phantom in Figure 4 on a cubic (◇), fcc (□), and bcc (○) lattice.

In Figure 6, the relative difference between the fcc/bcc lattice and the cubic lattice in represented energy as a function of sampling density is plotted.

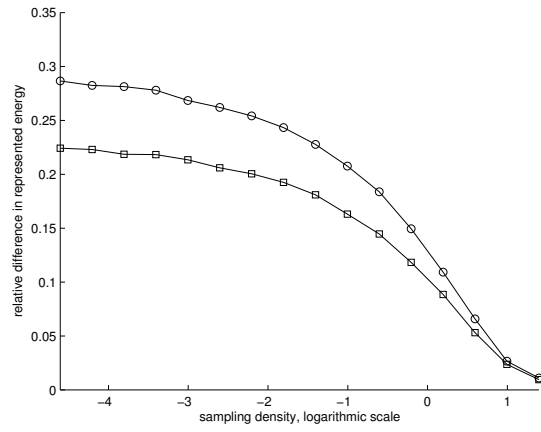


Fig. 6. The relative difference between the fcc (□)/bcc(○) lattice and the cubic lattice in represented energy as a function of sampling density when sampling the 3D Shepp Logan phantom in Figure 4.

IV. CONCLUSIONS

As expected, the fcc and bcc lattices are more efficient than the cubic lattice when it comes to sampling functions. It is known that if all frequencies should be represented when sampling a band-limited function, almost 30% fewer samples can be used when a bcc lattice is used compared to when a cubic lattice is used [1]–[3]. When sampling the non-bandlimited test function used here such that 2% of the relative energy is represented, then $1 - 0.55/0.64 \approx 14\%$ fewer samples can be used when a bcc lattice is used instead of a cubic lattice. For the Shepp-Logan phantom, the gain is almost 30% when the sampling density is low, see Figure 6. When the sampling density is high enough, then all frequencies are represented independent of the choice of sampling lattice. This can also be seen in Figure 6.

Figure 3 illustrates that both the effect of aliasing and the rotational dependency is lower on the fcc and bcc lattices compared to the cubic lattice on one-dimensional linear subsets of the three-dimensional Gauss-function used here.

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