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Reiner Lenz
Linköping University,
SE-60174 Norrköping, Sweden,
reiner.lenz@liu.se

Rika Mochizuki
NTT Cyber Solutions Laboratories,
Yokosuka-shi, 239-0847 Japan,
mochizuki.rika@lab.ntt.co.jp

Jinhui Chao
Dept. Information and System Engineering, Chuo University,
Tokyo, 112-8551 Japan,
jchao@ise.chuo-u.ac.jp

Abstract—We investigate several topics related to manifold-techniques for signal processing. On the most general level we consider manifolds with a Riemannian Geometry. These manifolds are characterized by their inner products on the tangent spaces. We describe the connection between the symmetric positive-definite matrices defining these inner products and the Cartan and the Iwasawa decomposition of the general linear matrix groups. This decomposition gives rise to the decomposition of the inner product matrices into diagonal matrices and orthonormal and into diagonal and upper triangular matrices. Next we describe the estimation of the inner product matrices from measured data as an optimization process on the homogeneous space of upper triangular matrices. We show that the decomposition leads to simple forms of gradients and Hessian matrices that are commonly used in optimization algorithms. Using the parametrization from the symmetric space ensures also that all intermediate estimates of the inner product matrix are symmetric and positive definite. Finally we apply the method to a problem from psychophysics where the color perception properties of an observer are characterized with the help of color matching experiments. We will show that measurements from color weak observers require the enforcement of the positive-definiteness of the matrix with the help of the manifold optimization technique.

I. INTRODUCTION AND OVERVIEW

A Riemannian manifold is a generalization of the common Euclidean space. Intuitively it looks locally like a Euclidean space but the metric, i.e. the way how angles and distances are measured, can change from one point in the space to the next. Traditionally the metric is given analytically and properties of the space are derived from its metric.

Recently there has been an increased interest in applying methods from Riemannian geometry in the analysis of measured data. Here the assumption is that the investigated space is equipped with a Riemannian geometry but the exact form of the metric is unknown and has to be estimated from measurements.

In this paper we will first show how the metric can be investigated in the framework of matrix groups. From

the general properties of the matrix group we derive a parametrization of the metric that is especially suited for the numerical estimation of the metric from measured data.

We will illustrate the usefulness of the technique by describing its application to the construction of a metric that describes the color perception of individual observers.

II. RIEMANNIAN GEOMETRY AND MATRIX GROUPS

A Riemannian Manifold is a differential manifold with an inner product on each of its tangent spaces (for a detailed description see one of the numerous textbooks such as [1]). One of the simplest examples of a Riemannian manifold is the unit circle where we define an inner product on each of its (one-dimensional) tangent spaces. For every point of the circle a small section of the circle around this point is very similar to a line segment but the circle as a whole is very different from a line. At each point we can define the length of a vector in the vector space given by the metric on the tangent line.

The inner product in a vector space (with a given basis) is specified by a symmetric positive definite (SPD) matrix S and in the following we will derive a parametrization of these matrices from the general theory of matrix groups. We start with the observation that for a general, quadratic, non-singular matrix M the matrix product $S = M'M$ is an SPD matrix (M' denotes the transpose of M). From the general theory of matrix groups it is well-known that there are two useful decompositions of matrices:

- **Iwasawa** Decomposition $M = KAN$
- **Cartan** Decomposition $M = K_1AK_2$

where K, K_1, K_2 are orthonormal matrices, A is a diagonal matrix and N is an upper triangular matrix with ones in the diagonal. For the Cartan decomposition we have for the symmetric matrices

$$S = M'M = K_2'AK_1'K_1AK_2 = K_2'A^2K_2'$$

and we find the parametrization of the symmetric matrices by diagonal and rotation matrices that is familiar from principal component analysis (PCA). The Cartan decomposition is also known as the polar decomposition. In the following we will mainly apply the Iwasawa decomposition with:

$$S = M' M = N' A K' K A N = N A^2 N'$$

We will write $U = AN$ and describe the symmetric matrix as a product of an upper triangular matrix with positive diagonal elements and its transpose. These upper triangular matrices form another manifold with its own tangent spaces (for more information see [2]). One of the main advantages of the Iwasawa decomposition originates in the connection between the matrix group, its tangent spaces and differential operators. From the definition of the diagonal matrices A we find that these matrices define a group under matrix multiplication and that this group depends on N parameters. We denote by X_{nn} the $N \times N$ matrix with a 1 at diagonal position (n, n) and zeros everywhere else. Since the matrices A have positive elements in the diagonal we find that every matrix A has the form $e^{\sum_n a_n X_{nn}}$. For an index pair (m, n) , $m < n$ we define the $N \times N$ matrix X_{mn} which has a 1 at position (m, n) and zeros everywhere else. For these matrices we see that $e^{a X_{mn}}$ has ones in the diagonal, the entry a at position (m, n) and zeros everywhere else. Now consider a vector ξ and the function $\xi' e^{a X_{mn}} e^{a X_{mn}} \xi$. For $m = n$ we have

$$f_{nn}(a, \xi) = \xi' e^{a X_{nn}} e^{a X_{nn}} \xi = e^{2a} \xi_n^2 + \sum_m x i_m^2$$

where the sum is over all m except $m = n$. From this we get

$$\frac{\partial f_{nn}(a, \xi)}{\partial a} \Big|_{a=0} = 2\xi_n^2$$

For $m < n$ we have by the same construction

$$\frac{\partial f_{mn}(a, \xi)}{\partial a} \Big|_{a=0} = 2\xi_m \xi_n$$

Sometimes we will also use the parametrization of the matrices U by the matrix exponential $U = e^X$ where X is an upper triangular matrix. Using the definition of the matrix exponential we find the following (second order) approximation:

$$\begin{aligned} S &= U' U \\ &\approx \left(I + X + \frac{X^2}{2} \right) \left(I + X' + \frac{X'^2}{2} \right) \\ &\approx I + X + X' + \frac{X^2 + X'^2 + 2XX'}{2} \end{aligned} \quad (1)$$

As a result we see that we can identify a symmetric matrix with the AK or the AN part of the Cartan or Iwasawa decomposition. The corresponding classes of the equivalence relation form a factor or homogenous space of the group of general linear matrices.

III. ESTIMATION

In our application we have a set of measurement vectors $\xi_k, k = 1, \dots, K$ that have, by definition, length one in the unknown metric (a traditional statistical solution of the estimation problem in the presence of noise can be found in [3]). They define thus K equations $\xi_k' S \xi_k = 1$. Writing out one of these equations in detail we find that

$$\sum_{m=1}^N \sum_{n=1}^N \sigma_{mn} \xi_{mk} \xi_{nk} = 1$$

where the σ are the elements of S , ξ_{mk}, ξ_{nk} are the elements of ξ_k and N is the dimension of the Riemannian manifold. Collecting the measurement values ξ_{mk} in the data matrix D (of size $K \times \frac{N(N+1)}{2}$) and the elements σ_{mn} in the coefficient vector Σ we see that the condition for the matrix elements are given by the matrix equation $D\Sigma = \mathbf{1}$ where $\mathbf{1}$ is the vector of length K where all elements have value one. Solving this equation (for example using a minimum-mean-squared-error algorithm) will provide an estimate of the inner product matrix S . The problem with this approach is that it cannot guarantee that S is an SPD-matrix.

Instead of the using this simple MMSE-approach we will in the following show how to use the Iwasawa decomposition and its exponential form to find solutions that are by construction always SPD-matrices. The idea is to construct a sequence of matrices $X_j, j = 1, \dots, J$ such that $U_j = e^{X_j}$ and $U = U_J \dots U_1$ converges to a solution of the equations

$$\xi_k' S \xi_k = \xi_k' U' U \xi_k = 1$$

This approach leads to an iterative process where the basic update step has the following structure:

- 1) Collect the modified data $U_J \dots U_1 \xi_k$ at current step J into the data matrix D_J
 - 2) Define an energy- or quality function $Q(D_J, X)$
- Examples are the estimator

$$D_J' e^{X'} e^X D_J = 1$$

or a mean squared error

$$\sum_k \left(\xi_k^{J'} e^{X'} e^X \xi_k^J - 1 \right)^2$$

- 3) Find a solution X_{J+1} using approximation (1)
- 4) Compute $U_{J+1} = e^{X_{J+1}}$.

Solving for X has the advantage that the positivity constraint is automatically satisfied by the exponential transform. The details of this solution depends on how we define the quality function $Q(D_J, X)$ and what algorithm we use to find the approximative solution X_J . From equation (1) we see how we can use the approximation of the matrix exponential to find an approximative solution. From the discussion in connection with the Iwasawa decomposition we find partial derivatives $\frac{\partial f_{mn}(a, \xi)}{\partial a} \Big|_{a=0}$ if we write the quality function in the group theoretical parametrization.

IV. COLOR SPACES

Color spaces are important examples of Riemannian manifolds. They were used as examples by Riemann himself when he introduced what is now known as Riemannian geometry (see [4] and [5]).

Here we denote a color, i.e. a point in color space by $\mathbf{x}$ and we use a coordinate system to describe the color with the coordinate vector $\xi(\mathbf{x})$. Typical examples of such coordinate systems are the CIE systems CIEXYZ, CIELAB and CIELUV, but others like one of the RGB variants or a system adapted to a given measurement or display device may also be used. One method used to measure the color perception of an observer is color matching (see [6] for a description and [7] for a critical assessment of the approach). At the beginning of a color matching experiment the observer is shown two identical color patches $\hat{\mathbf{x}}_b, \hat{\mathbf{x}}_c$ side-by-side on a display. Then the one to be compared, $\hat{\mathbf{x}}_c$ is changed (usually along a line in the coordinate space $\xi(\mathbf{x})$ until the observer sees a difference between the base patch and the modified patch for the first time. This new, modified color $\mathbf{x}_\Delta$ defines a just-noticeable difference (jnd) for the observer. In this way every experiment produces a difference vector ξ_Δ in the coordinate system used. For a given base color $\hat{\mathbf{x}}$ we denote the resulting difference vectors of K matching experiments by $\xi_k, k = 1, \dots, K$ and if we want to incorporate the base color we will write $\xi(\hat{\mathbf{x}})_k, k = 1, \dots, K$. By definition we have

$$1 = \xi(\hat{\mathbf{x}})_k' S(\hat{\mathbf{x}}) \xi(\hat{\mathbf{x}})_k$$

where $S(\hat{\mathbf{x}})$ is the matrix to be estimated.

The basic idea behind color matching is very simple but for a successful application a number of difficulties have to be taken into account. These problems are addressed in psychophysics and we only mention them, but we don't go into detail here. First one has to control the experimental conditions very carefully. Some important factors are the adaptation state of the observer, the viewing geometry and the role of the background on which the patches are shown. This is the reason why the results of single experiments have a high variability and therefore they have to be repeated several times; in the classical experiments by MacAdam they were repeated 50 times. Also, by definition, the differences between the colors will be very small since they define the limit of the color resolution of the observer. This can lead to numerical problems in the estimation. Finally we are interested in the color perception of color-weak observers. Such an observer is not colorblind but he/she has a color vision that is significantly different from a color normal observer. This means that the observer has a significantly reduced discrimination along a direction in color space. For colorblind observers the matrix $S(\hat{\mathbf{x}})$ will be no longer of rank three but of rank one or two. For all of these reasons it is important to have a robust estimator that can produce valid estimates of the metric from noisy data.

V. AN ILLUSTRATION

In our experiments we are interested in the color perception of color normal and color weak observers [8], [9]. The measured jnd data consists of experiments done by two individual observers and by the mean results obtained from 45 observers. The measured colors are described in the CIEXYZ coordinate system. The color matching experiments were done for 10 base points and for every base point the color was varied in 14 different directions CIEXYZ space.

We mentioned above that the numbers describing the differences will be very small in the CIEXYZ system. We therefore pre-multiply all vectors with a factor of 100. For the measurement sequences we calculated first the linear estimator Σ from the equation $D\Sigma = 1$ using the backslash operator in Matlab. From Σ we computed the symmetric matrix S_0 and its eigenvectors and eigenvalues. We found that the last eigenvalue was often very small or negative. In the singular value decomposition of S_0 we replaced the negative eigenvalues by a small positive value and computed the new symmetric matrix S_1 using the eigenvectors of the original matrix S_0 . This matrix and its Iwasawa decomposition was then used as start value in the iterative method outlined above. Figure V shows one example of an estimated ellipsoid. It shows the ellipsoids and the 14 measurement vectors used to define it (note that all data are magnified by a factor 100).

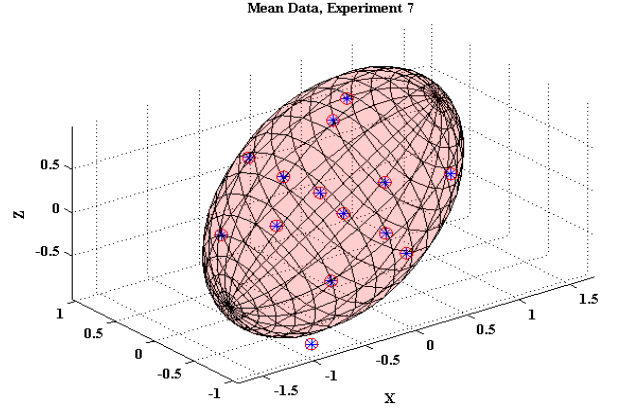


Fig. 1. Ellipsoids and measurements for mean data

In Figure V we show two ellipsoids illustrating the differences between the jnd's of one of the observers and the ellipsoid computed from the mean measurement vectors. It shows that in that region of color space the color discrimination of the observer was significantly lower than the average discrimination of the color normal observers.

As an example where the initial estimator based on the backslash operator produced a matrix with a negative eigenvalue we show one experiment with the color weak observer. One of his measurement series resulted in the estimated matrix

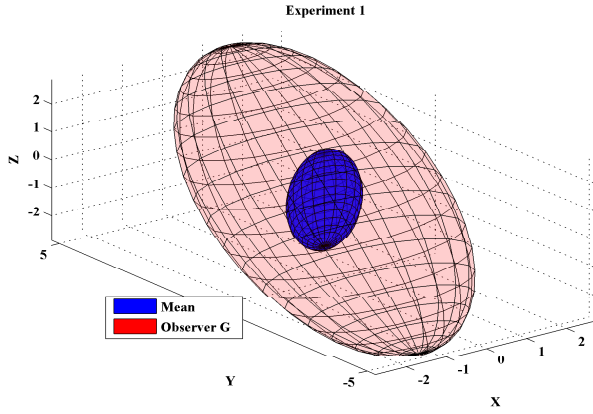


Fig. 2. Ellipsoids for mean and one observer (100x enlarged)

$$S_0 = \begin{pmatrix} 0.4864 & 0.3731 & -0.4596 \\ 0.3731 & 0.0892 & -0.2463 \\ -0.4596 & -0.2463 & 0.3058 \end{pmatrix}$$

with lowest eigenvalue -0.0457 . After the optimization iterations the resulting matrix was given by

$$S = \begin{pmatrix} 0.5167 & 0.2062 & -0.2149 \\ 0.2062 & 0.1142 & -0.1518 \\ -0.2149 & -0.1518 & 0.2259 \end{pmatrix}$$

which is singular. The singularity indicates that the observer cannot discriminate color variations along that line and that he is therefore probably colorblind.

VI. CONCLUSIONS

The main goal of this contribution was to show that the standard Iwasawa decomposition of the general linear group leads to a useful parametrization of symmetric positive-definite matrices. The parameters have an intuitive interpretation as scaling and translation operations and derivatives based on these operations can easily be computed. As an application we choose the investigation of color perception measurements and their application in the investigation of color spaces. We showed how the parametrization automatically lead to positive (or semi-positive) estimation matrices.

The detailed investigation of the influence of the noise in the measurements lies outside the scope of this paper but it is of fundamental importance since the goal of the measurements is the determination of the resolution limit of the observer.

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